J. Fluid Mech. (1975), *vol.* 67, *part* 2, *pp.* 377–396
Printed in Great Britain

Finite amplitude analysis of a flow–structure interaction problem

By A. O. ST HILAIRE† AND P. G. VAIDYA‡
Department of Mechanical Engineering, Tufts University,
Medford, Massachusetts

(Received 26 December 1973)

The interaction between a vibrating structure and an unsteady potential flow-field disturbance induced by the motion of the structure itself is investigated, and is shown to be a significant source of both nonlinear excitation and nonlinear dissipation. An approximate analysis, based on small nonlinear disturbance theory, is presented of the forces that influence the characteristic behaviour of self-excited harmonium reeds vibrating at finite amplitudes. It is demonstrated that the ideas brought forth by this example can be generalized to apply to other flow-induced vibrating systems, regardless of the excitation mechanism, provided that certain basic assumptions about the flow can be made. For the case of the harmonium reed, it is shown that, taken by itself, an account of the feedback forces arising from induced higher-order unsteady disturbances in the surrounding potential flow field is sufficient for predicting the net nonlinear dissipative force that eventually causes the reed to reach and maintain a finite limiting amplitude. In particular, it is demonstrated that the nonlinear energy drain from the motion of the reed is a consequence of the net effect of the higher-harmonic disturbances that are generated near the structure.

A result of the analysis is the development of a functional dependence of the interactive forces on the system geometry and the flow velocity. One of the advantages of obtaining a functional expression is the ability to carry out parametric studies in the context of vibration and noise control.

1. Introduction

The success of an analysis of any problem in the area of noise and vibration control often depends on a precise determination of the source characteristics, including an identification of the inducting mechanism. Once the exciting mechanism is known, the investigator is better equipped to deal with the vibration problem. If there is acoustic radiation associated with a flow-induced vibrator, for example, it may even be possible to gain insight into the inherent characteristics of the radiating pressure field, based on the knowledge of the associated flow field at the source. As a consequence, such a source identification is a significant part of the effort in noise-control research.

† Present address: United Aircraft Research Laboratories, East Hartford, Connecticut.
‡ Present address: R. W. Herrick Laboratories, Purdue University, Lafayette, Indiana.

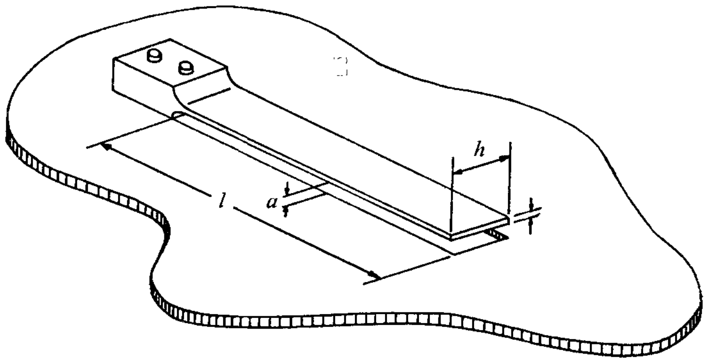

FIGURE 1. Schematic diagram of the harmonium reed and shallot plate.

The present work concerns the nonlinear fluid dynamics of a flow-induced structural-vibration problem. Specifically, a study of the nonlinear fluid–elastic phenomena associated with the harmonium reed is presented, along with a discussion of the implications of the results. Although this is a relatively simple device, the ideas and results that are brought forth are of fundamental importance, and can be generalized to include other systems which may be more complex.

In a paper published by St Hilaire, Wilson & Beavers (1971, hereafter referred to as SWB), an analysis was presented that described qualitatively the mechanism responsible for inducing the self-excited motion of a vibrating metal reed (shown schematically in figure 1) in the presence of fluid flow. It was shown that the pressure forces developed in the unsteady potential flow past the reed excite the reed motion. A linearized form of the equations that represent an assumed unsteady potential flow about the oscillating structure was analysed, and the results predicted certain observed phenomena, such as the exponential amplitude growth undergone by the reed in the initial stage of excitation. The analysis also provided an accurate prediction of the actual magnitude and trend of the linear fluid–elastic forces, as functions of the volume flow past the reed assembly.

Although the SWB analysis permits the successful development just outlined, its application is restricted to amplitudes of the motion that are relatively small. An effective means of analytically predicting the magnitude of the system motion beyond the limited range of amplitudes allowed by linear analysis is not only of academic interest, but also of broad technological interest, especially in certain applications concerning the prevention or control of vibration and noise.

A linear analysis of the fluid–elastic interaction yields useful information about the initial stage of the oscillator motion; to be able to estimate analytically the limit of excitation, however, requires a knowledge of what the significant dissipative and excitative nonlinear forces that act on the excited structure are. The principal focus of this paper is on those features that become progressively more important as amplitudes of vibration larger than those allowed by linear analysis are generated. Although there are several possible sources of nonlinear energy dissipation that could be considered in the analysis of the nonlinear interactive forces, only one source is examined in the following sections. This source is a consequence of the unsteady finite amplitude disturbances that are generated within a non-dissipative potential flow field by the structure's own motion. Although the interaction of the various components of the flow-field disturbance within the fluid is intricate, the nonlinear effects that arise will be accounted for as simply as possible. The desired objective of demonstrating the significance of this source of energy dissipation for flow–structure interaction problems in general is achieved by studying the case of a self-excited harmonium reed. It is shown that it is sufficient, at least for the case of the harmonium reed, to consider only those effects arising from interactions of the type just described in order to be able to predict the degree of reed excitation in terms of the nonlinear forces acting on the reed. Comparisons with experiment for two reed configurations are offered to substantiate the analysis.

2. Background

2.1. *Introduction*

There are a number of mechanical systems that display the characteristics of self-sustained oscillation. In some of these, the oscillations are produced by some feedback mechanism, the energy required to sustain the motion being derived from a time-independent energy source. An example of such a device is the metal reed, which obtains its energy of oscillation from the surrounding flow field.

The harmonium reed is one of a variety of sound sources that have been developed empirically through the years to satisfy the criterion that they be efficient producers of sound. Although the harmonium reed is simple in construction, the precise analysis of the forces that influence its motion requires a consideration of a large number of factors, both linear and nonlinear. The fluid–elastic forces that have been found both to excite the reed motion (SWB) and, as shown in § 3, to cause the reed to reach and maintain a finite limiting amplitude are very complicated; an attempt, therefore, at their precise analysis would not be of practical value. In order to render the problem amenable to approximate analysis, it is necessary to make certain basic assumptions about the geometrical configuration and the surrounding flow region.

A linear analysis of the interaction between the harmonium reed and the flow field surrounding it has already been made. As a consequence, the assumptions that are common to both the previous and present analyses of the problem are only briefly described, since a detailed description is available in SWB.

2.2. *Reed-assembly geometry*

The harmonium reed is essentially a small flat beam riveted at one end to a support plate hereafter called the 'shallot'. A schematic diagram of the assembly is shown in figure 1. The cantilevered portion of the reed is shown as having its equilibrium position a uniform distance a above the upstream side of the shallot, thus forming two gaps of area al. The rectangular opening in the shallot immediately beneath the reed, being slightly larger than the reed itself, allows the reed

Reed	f_0 (Hz)	t (cm)	h (cm)	a (cm)	l (cm)
A	165	0·022	0·577	0·035	3·80
B	225	0·030	0·460	0·020	4·20

TABLE 1. Reed characteristics

to vibrate freely as the vibrational amplitude becomes larger. Typically, for such a reed configuration, it is reasonable to assume that $a \ll h \ll l$. This allows the flow near the reed to be considered nearly two-dimensional. This assumption, furthermore, provides a scheme for simplifying the expressions that are obtained in the analysis.

In order to study the growth of the reed vibration in the laboratory, two reeds were attached to appropriate shallots. The dimensions of the two reed assemblies are given in table 1, which also includes the natural frequency f_0 for each reed. The study of the reed motion included the case where the reeds were modified by the addition of small masses to their tips. The masses were used to reduce the vibration frequency of the reeds without changing their geometry.

2.3. *Reed dynamics*

When any structure with an inclination towards flow-induced oscillation has its position of static equilibrium (which includes any displacement resulting from an imposed constant drag force) altered slightly under favourable conditions of flow, even by the most infinitesimal disturbance, such as a shedding vortex, it begins to vibrate with a growing amplitude. It has been observed, for example, that the self-excited oscillation of the harmonium reed occurs when the pressure difference between the two reservoirs on either side of the reed assembly is large enough that the periodic flow disturbances introduced by the reed into the surrounding potential flow field feed enough energy back to the reed to offset the dissipation effect of the inherent linear damping of the system. As a result, an energy feedback loop is set up between the flow field and the structure, so that the amplitudes of both the structure's motion and the flow disturbance become increasingly large. A theoretical calculation based on linear analysis gives rise to an unstable situation; that is, it allows the regenerative cycle just described to proceed without limit. In reality, however, small disturbances only grow until either a nonlinear limiting mechanism appears or there is structural failure. For the case of the harmonium reed, it has been observed that the initial exponential growth rate does slow down because of an undefined nonlinear dissipating mechanism, until the motion eventually levels off at and maintains a certain limiting amplitude.

2.4. *Nonlinear forces*

Most authors working in nonlinear mechanics seem to agree that the identification of the sources of nonlinear forces is often as difficult as the prediction of their strength. For the case of a flow-induced structural-vibration problem, such as the self-excited harmonium reed, it can be safely assumed that only those nonlinear forces that are in phase with the velocity of the structure significantly influence its fundamental motion. This is because the inertial and stiffness elements of the structure are much less sensitive than the damping term to fluid–elastic interactions, and inclusion of these effects in the analysis would only yield a slight refinement of the main results that are sought in the present work. Currie, Hartlen & Martin (1972) have considered the effects of the nonlinear spring element in their phenomenological study of the vortex-induced oscillations of a cylinder in free-stream flow. Although their work was partly empirical, their results provided a more complete understanding of the overall behaviour of the oscillating cylinder. For the present problem, an analytical approach to the development of the nonlinear spring and mass elements of the reed arising from flow–structure interactions is possible simply by retaining the imaginary part of the nonlinear damping force (§3), and decomposing it into its inertial and stiffness components. Such a development is not included here, however.

There are several sources from which nonlinear contributions to the system damping may arise. In their study of oscillating flat cantilevered beams, Baker, Woolam & Young (1967) found that the aerodynamic drag associated with large amplitude motion gives rise to a nonlinear force that is proportional to the amplitude squared and in phase with beam velocity; observations indicate, however, that a stronger dependence of the nonlinear force on amplitude prevails for the present configuration. Another source of nonlinear damping concerns the internal friction of the structure. Many investigators (e.g. Lazan & Goodman 1961; Crandall, Khabbaz & Manning 1964) have proposed, using empirical formulations, that the associated nonlinear force is proportional to the amplitude raised to the seventh power. Because such a force is of very high order, its effects are not considered here. The analysis that follows shows that the nonlinear fluid–elastic forces that are generated from flow-field disturbances give an adequate description of the behaviour of the harmonium reed at finite amplitudes of vibration without account being taken of other sources of nonlinear dissipation.

2.5. *Flow model*

The perturbations that are exerted by the reed on the fluid can be taken into account by assuming that the flow is a superposition of steady potential flow and higher-order, unsteady effects. The reed assembly, as shown in figure 1, acts both as a dipole and a monopole source when the reed becomes excited. Once the reed was set in motion in the laboratory, it was observed that its vibrations were simple harmonic, and yet it was able to radiate a multi-harmonic acoustic signal. This situation naturally suggests the presence of induced higher-order pressure fluctuations in the surrounding flow field, some of which are capable of radiating acoustically. In consequence, disturbance terms up to at least the third order (amplitude of the motion cubed) are included in the analytical model. The objective of this work is to show that the nonlinear effects associated with the higher-harmonic fluctuations in the fluid very near the reed become significant sources of nonlinear excitation and dissipation at finite amplitudes of vibration.

There are two relationships necessary to execute the solution of fluid–elastic problems of the type under discussion. The first relationship is a constraint equation which allows the flow disturbance to be related to the position of the vibrating structure. The second is an expression for the pressure field that allows computation of the components of the fluid–elastic force which most significantly affect the structure's behaviour.

An unsteady potential flow analysis yields the difference between the upstream and downstream reservoir pressures in terms of the flow variables as

$$\frac{\Delta p_0}{\rho} = \frac{Q^2}{2\pi^2 l^2 a^2} - \frac{Q\dot{a}}{\pi^2 l a}\ln\left(\frac{h}{a}\right) + \frac{\dot{a}^2}{2\pi^2}\left[\left(\ln\frac{h}{a}\right)^2 - 4\right] + \frac{\dot{Q}}{\pi l}\left[2 + \ln\left(\frac{l^2}{h(2ha)^{\frac{1}{2}}}\right)\right] - \frac{h\ddot{a}}{\pi}\left[2 + \ln\left(\frac{l^2}{2h^2}\right)\right], \tag{1}$$

where Q is the total volume flow through the two gaps, each of constant length l and of time-varying width a. Equation (1) represents the constraint relating any changes in Q to corresponding changes in a for the oscillating reed.

The objective of the analysis is to calculate the component of the fluid–elastic force which is in phase with the reed velocity. This force component is derived from the pressure difference Δp across the reed,

$$\begin{aligned}\frac{\Delta p}{\rho} = \frac{\Delta p_0}{\rho} - \frac{\dot{Q}}{2\pi l}\left[4 + \ln\left(\frac{l^4}{4h^2(x_0^2 - x^2)}\right)\right] \\ + \frac{\ddot{a}}{\pi}\left\{x \ln\left(\frac{\frac{1}{2}h - x}{\frac{1}{2}h + x}\right) - \frac{h}{2}\ln\left(\frac{h^2}{4} - x^2\right) + h\left[2 + \ln\left(\frac{l^2}{2h}\right)\right]\right\} \\ - \frac{1}{2}\frac{Q^2}{\pi^2 l^2}\left[\frac{x^2}{(x^2 - x_0^2)^2} - \frac{4}{h^2}\right] - \frac{Q\dot{a}}{\pi^2 l}\left[\frac{x}{x_0^2 - x^2}\ln\left(\frac{\frac{1}{2}h + x}{\frac{1}{2}h - x}\right) + \frac{4}{h}\right] \\ - \frac{\dot{a}^2}{2\pi^2}\left[\left(\ln\frac{\frac{1}{2}h + x}{\frac{1}{2}h - x}\right)^2 - 4\right].\end{aligned} \tag{2}$$

The variable x indicates the position across the width of the reed surface on the upstream side. The points $x = \pm x_0$, where $x_0 = \frac{1}{2}(h + a)$, are the positions of the two line sinks formed by the two gaps between the reed and the shallot relative to the reed centre-line (SWB). It is noted that the above two equations contain an $\dot{a}^2$ term, which, being of higher order, was neglected in SWB. A detailed account of (1) and (2) is available in a report by St Hilaire (1971), as well as in SWB.

3. Analysis of fluid–elastic forces

3.1. *Flow as a function of structure displacement*

As was stated in the previous section, it is assumed at the outset that the only important source of nonlinear rates of change of energy experienced by the reed is the existence of the higher-order flow fluctuations that are introduced by the reed motion itself into the surrounding steady potential flow field. It is shown in this section that, when the amplitudes of these fluctuations become large, the net effect of the associated higher-order forces that act on the structure is dissipative, and that, as the amplitudes continue to grow, these forces eventually become significant enough to curb the regenerative process predicted by linear analysis.

Since the present analysis includes higher-order unsteady effects, it follows that writing out a proper expression that gives the induced unsteady flow perturbation as a function of the reed displacement is more complicated than in the linear case. Certain justifiable assumptions are made, however, to keep the expression as simple as possible without sacrificing the important details. Rather than attempting to obtain such an expression directly from (1), the desired expression is assumed to be of a form that is based on the following arguments.

In finite amplitude wave or disturbance analyses, the phenomena associated with the interactions of the primary disturbances can no longer be neglected. These interactions give rise to higher-frequency, higher-order fluctuations in the flow field near the structure. A consequence of these higher-order fluctuations is that the oscillating structure becomes subjected to the corresponding higher-harmonic forces that are generated. Therefore, it would appear that the oscillator would tend to vibrate at several harmonics whose frequencies are integral multiples of the frequency of its fundamental motion. However, because of the high selectivity of the type of oscillators under discussion, such as the harmonium reed, it is appropriate to assume, with confidence, that the phenomenon of resonance prevails.† This allows the suppression of higher-harmonic terms in the expression for the oscillator motion. Such an assumption, which is borne out experimentally, has a good deal of practical value, in so far as it results in a simpler and more straightforward analysis. It is crucial, however, to distinguish between the relative significance of higher-harmonic oscillator activity and the higher-harmonic motion of the flow disturbance. For, although the former can be neglected with some justification, the latter must be taken into account in order to succeed in determining the nonlinear fluid–elastic forces that are associated with the motion of the oscillator. Not only is the retention of these higher-harmonic fluctuations useful from a fluid-dynamic standpoint, but a reasonably good qualitative knowledge of these fluctuations is necessary in order to be able to estimate analytically, at least in part, the multi-harmonic content of the acoustic signal radiating from the device.

It is also appropriate to assume that any higher-mode activity of the reed or similar structure can be safely neglected. If the higher modes, with frequencies that are far from being integral multiples of the fundamental, persisted at appreciable amplitudes, the motion would be complex, thereby indicating that the reed gives off a discordant acoustic signal, contrary to what is observed to emanate from a musical instrument. In support of this inference, the higher-order vibrational modes of the beam were not observed when the dynamic responses from the strain gauges were monitored. They are not present in any significant magnitude for two reasons. First, because the flow is nearly uniform over the length of the reed, there is no significant component of the forces acting on the reed in these modes, and second, because higher modes have higher modal

† Once a feedback loop is established between a self-excited structure and the flowing fluid, one might think of the structure as setting up its own forcing function at the structure's own frequency. Furthermore, because of the expected high selectivity (weak damping) of self-excited systems, one can safely say that this is the only frequency that is generated to any significant degree.

impedances, they are more strenuous to produce. Consequently, the reed motion is nearly periodic, and almost entirely due to the fundamental.

A convenient method of explicitly extracting information about the volume flow as a function of the size of the gap between the reed and its supporting structure is to assume that these variables have the following form:

$$a = a_0 + a', \quad Q = Q_0 + Q', \tag{3}$$

where a_0 corresponds to the undisturbed position of the reed away from the shallot for a given Q_0 prior to the onset of self-excited vibratory motion and Q_0 is the steady flow (which was regulated by a calibrated orifice) past the reed assembly. The quantities a' and Q' represent the respective time-varying deviations from these constant values as the reed is set in motion.

On the basis of the arguments just presented, the expression for a' may be written as

$$a' = a_1 e^{-i\omega t}. \tag{4}$$

One of the possible ways of accounting for the higher-order effects in the flow is by breaking down into their harmonic components the nonlinear terms that would appear in the expression for Q'. This approach seems to be the most natural for enabling a description of the sources of nonlinear dissipation and excitation. Therefore, the following Fourier series expansion for Q' is assumed:

$$Q' = Q_s + \sum_n Q_n \, e^{-in\omega t}, \tag{5}$$

where Q_s represents the non-oscillatory component of the flow perturbation, and the coefficients of the series are power-series expansions in a_1. The justification for this eclectic choice of the form of Q' lies in the successful methodical analysis to which it leads, while at the same time preserving a grasp of the physics of the problem.

It is noted that, since the motion of a weakly damped oscillator is quasi-steady, all amplitudes appearing in (4) and (5) can be regarded as constants with respect to time, thereby greatly simplifying the analysis. On the other hand, because the problem is nonlinear, the evaluation of the assumed time-independent coefficients Q_n of (5) is slightly complicated by the interdependence of the various coefficients within the Q_n.

When writing out (5) in terms of a_1, it is useful to break up the coefficients of each harmonic component in such a way as to be able to distinguish between two types of mechanisms that contribute to the fluid–elastic interaction. The first mechanism has to do with the self-interaction of the first-harmonic component of the unsteady flow perturbation, as well as its interaction with the simple harmonic motion of the structure. This interaction gives rise to higher-order, higher-harmonic fluctuating components (including a non-oscillatory component) within the flow, as well as creating higher-order contributions to the first-harmonic component of the flow disturbance. The second mechanism accounts for the subsequent mutual interaction of the various components of the flow disturbance (again including the non-oscillatory component) within the flow near the structure. This mutual interaction of the various components of the disturbance, as well as the self-interaction of the higher-harmonic components, provides still more higher-order contributions to all the components of the flow disturbance.

For simplicity, the two mechanisms which contribute higher-order adjustments to the fundamental (first harmonic) disturbance will hereafter be referred to as mechanism I (self-interaction effects) and mechanism II (mutual-interaction effects), respectively. The purpose of accounting for each of these mechanisms separately is to see whether the generation of higher-harmonic disturbances in the fluid is a significant enough drain of energy away from the vibrating system to counteract any nonlinear excitation that may arise from mechanism I. Such knowledge would certainly be helpful in explaining how the reed reaches a limiting amplitude.

By taking into account the distinguishing features of these two types of interaction, (5) can be written in terms of the displacement amplitude a_1 as

$$Q' = \nu a_1 e^{-i\omega t} + \sum_k \sum_{n=1}^{k} [\mu_k + (\alpha_{k,\,2n} + \beta_{k,\,2n})\, e^{-2ni\omega t}]\, a_1^{2k} + \sum_k \sum_{n=1}^{k+1} (\alpha_{k,\,2n+1} + \beta_{k,\,2n+1})\, e^{-(2n-1)\,i\omega t} a_1^{2k+1}, \tag{6}$$

where the coefficients $\alpha_{k,j}$ and $\beta_{k,j}$ are associated with mechanisms I and II, while the coefficients ν and μ_k are related to the linear and non-oscillatory parts of the flow disturbance, respectively.

As written, (6) is independent of the particular geometry, and therefore should apply, in theory, to any flow-induced structural vibrations where higher-harmonic activity and higher-frequency mode shapes of the structure's motion have been neglected. The subscripted coefficients appearing in (6) can all be written as functions of the configuration geometry, the flow velocity and ν; therefore, if a potential flow field can be formulated and an analytical or experimental value of ν specified for a given situation, (6) should be applicable, regardless of the exciting mechanism. As such, (6) can be used to determine whether or not the nonlinear interaction between a potential flow field and a vibrating body is an important source of energy dissipation. If this is so, this approach would allow at least a partial analytical prediction (in a qualitative sense) of the large amplitude response of a given system; that is, this approach would reveal explicitly the functional relations which govern the response of the structure. In addition, from an acoustic point of view, (6) is also useful as an important first step towards any attempt at estimating the resulting sound radiation in terms of the characteristics of the source for flow-induced structures that radiate acoustically.

It might be interjected at this point that, despite the elaborate form of the equation, only a few terms are needed to arrive at a reasonable analytic description of a flow–structure system, as demonstrated in the following sections for the reed. Furthermore, should any higher-harmonic activity of the structure be included in the analysis, there would be no explicit change in the form of (6). The resulting change would appear implicitly, since the coefficients μ_k and $\beta_{k,j}$

would become functions of the oscillator impedance. To include higher modal activity of the structure, on the other hand, would require rewriting (6) to include new frequency terms.

3.2. *Evaluation of the coefficients*

The coefficients of (6) can be evaluated upon substituting (3) into (1), expanding in a power series and, with the aid of (4) and (5), breaking up the resulting expression into its various harmonic components. In order to achieve the desired objective of analytically describing the nonlinear flow–structure interaction of the harmonium reed, it is sufficient to retain only those terms up to third-order deviations (a_1^3) in the expanded form of (1). Correspondingly, the non-oscillatory and first two harmonic components of (6) are all that are required for the present analysis. As a result, the equations that are relevant to the problem of the oscillating harmonium reed are

$$a' = a_1 e^{-i\omega t} \tag{7}$$

and, from (6),

$$Q' = Q_s + Q_1 e^{-i\omega t} + Q_2 e^{-2i\omega t}, \tag{8}$$

where

$$Q_s = \mu_1 a_1^2,$$

$$Q_1 = \nu a_1 + (\alpha_{1,1} + \beta_{1,1}) a_1^3, \quad Q_2 = (\alpha_{1,2} + \beta_{1,2}) a_1^2.$$

When (7) and (8) are substituted into (1) with the aid of (3) and only the linear terms in the perturbation quantities are retained, we find that

$$\tilde{\nu} = \frac{\nu a_0}{Q_0} = \frac{1 - (h/\pi a_0 q^2)\,[2 + \ln(l^2/2h^2)] - (i/\pi q)\ln(h/a_0)}{1 - (i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}}, \tag{9}$$

where q is the dimensionless volume flow represented by $Q_0/\pi\omega l a_0^2$. It can be seen from (9) that, if the frequency of the gap oscillation is small, the fractional change in the volume flow is the same as the fractional change in gap size. On the other hand, at higher frequencies, the flow perturbation is reduced and lags behind the gap perturbation on account of the inertia of the fluid.

By again combining (7) and (8) with the constraint equation, and this time retaining the terms proportional to a_1^3, we arrive at

$$\tilde{\alpha}_{1,1} = \frac{\alpha_{1,1} a_0^3}{Q_0} = \frac{\frac{3}{2}[1 - \mathrm{Re}(\tilde{\nu})] - \frac{3}{4}\tilde{\nu} + \frac{1}{2}|\tilde{\nu}|^2 + \frac{1}{4}\tilde{\nu}^2}{1 - (i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}}$$
$$- \frac{(i/\pi q)\{\frac{3}{8} + \frac{1}{4}\ln(h/a_0) - \frac{1}{4}\tilde{\nu}^*[1 + \ln(h/a_0)] - \frac{1}{8}\pi[\frac{3}{2}\tilde{\nu} - \mathrm{Re}(\tilde{\nu})]\}}{1 - (i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}}, \tag{10}$$

where an asterisk indicates a complex conjugate. The evaluation of μ_1 and $\alpha_{1,2}$ proceeds in a similar manner by retaining terms that are proportional to a_1^2. We find that

$$\tilde{\mu}_1 = \mu_1 a_0^2/Q_0 = -\tfrac{3}{4} + \mathrm{Re}(\tilde{\nu}) - \tfrac{1}{4}|\tilde{\nu}|^2 \tag{11}$$

and

$$\tilde{\alpha}_{1,2} = \frac{\alpha_{1,2} a_0^2}{Q_0} = \frac{-\frac{3}{4} + (i/2\pi q)[1 + \ln(h/a_0)]}{1 - (2i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}}$$
$$+ \frac{\tilde{\nu}\{1 - (i/2\pi q)[\ln(h/a_0) + \frac{1}{2}\pi]\} - \frac{1}{4}\tilde{\nu}^2}{1 - (2i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}}. \tag{12}$$

The contribution of the $\dot{a}^2$ term which appears in (1) is not included in (11) and (12) because, for values of the volume flow that excite the reed motion, the $\dot{a}^2$ contribution was found to be approximately three orders of magnitude smaller than any of the other contributing terms.

Having completed the evaluation of the coefficients of the terms that are associated with mechanism I, we can now calculate the remaining coefficients appearing in (8). From the constraint equation, the expression for $\beta_{1,1}$ is

$$\tilde{\beta}_{1,1} = \frac{\beta_{1,1} a_0^3}{Q_0} = \frac{\tilde{\alpha}_{1,2}\{1 + (i/2\pi q)\,[\ln(h/a_0) - \pi]\}}{1 - (i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}} - \frac{\frac{1}{2}\tilde{\alpha}_{1,2}\tilde{\nu}^* + \tilde{\mu}_1[2 - \tilde{\nu} - (i/\pi q)\ln(h/a_0)]}{1 - (i/q)\{2 + \ln[l^2/h(2ha_0)^{\frac{1}{2}}]\}}. \tag{13}$$

The dependence of $\beta_{1,1}$ on μ_1 and $\alpha_{1,2}$ is indicative of the coupling that exists between the various components of the disturbance. By inspection, the constraint equation shows that $\beta_{1,2}$ is equal to zero. In fact, inspection of (6) shows that $\beta_{k,2n}$ is zero for all n when k equals 1.

When the frequency of oscillation is lowered, it is seen from (10)–(13) that the coefficients become progressively smaller. This observation supports the expected equality between the fractional change in volume flow and the fractional change in gap size for low-frequency motion.

3.3. *The fluid-elastic forces*

The pressure forces that influence the dynamical behaviour of the reed through the damping term can now be evaluated. The component of the unsteady force which is in phase with the reed velocity is obtained by first substituting (3) into (2), and by using (8) to replace Q' in terms of a_1. Upon collecting terms that are linear and cubic in a_1, of lowest order in a_0/h and in phase with $\dot{a}$, the in-phase pressure component δp can immediately be written down. In this form, δp is a function of the position on the reed surface. Finally, the average force per unit area $\overline{\delta p}$ is obtained by integrating δp over the reed surface:

$$\overline{\delta p} = \frac{1}{h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h} \delta p\, dx. \tag{14}$$

Equation (14) is written such that a positive $\overline{\delta p}$ acts towards the downstream reservoir. The resulting approximate expression for the average dimensionless force, which both excites the reed motion and causes it eventually to reach a finite limiting amplitude, is

$$\frac{-\overline{\delta p}}{\rho Q_0 \dot{a}/2la_0} = \frac{2}{\pi}\left[3 + \ln\left(\frac{l^2}{2h^2}\right)\right]\mathrm{Re}(\tilde{\nu}) - \left[\frac{q}{\pi a_0 h}\,\mathrm{Im}\{2(\tilde{\alpha}_{1,1} + \tilde{\beta}_{1,1}) + \tilde{\nu}^*\tilde{\alpha}_{1,2} + 2\tilde{\nu}\tilde{\mu}_1\} - \frac{2}{\pi a_0^2}\left[3 + \ln\left(\frac{l^2}{2h^2}\right)\right]\mathrm{Re}(\tilde{\alpha}_{1,1} + \tilde{\beta}_{1,1})\right] a_1^2. \tag{15}$$

It is convenient, for the sake of discussion, to rewrite (15) as

$$\frac{-\overline{\delta p}}{\rho Q_0 \dot{a}/2la_0} = \frac{-\overline{\delta p_1}}{\rho Q_0 \dot{a}/2la_0} + \frac{-\overline{\delta p_2}}{\rho Q_0 \dot{a}/2la_0}, \tag{16}$$

where the first and second terms represent the linear and nonlinear parts of the dimensionless pressure force, respectively.

Inspection of
$$\frac{-\overline{\delta p_1}}{\rho Q_0 \dot{a}/2la_0} = \frac{2}{\pi}\left[3+\ln\left(\frac{l^2}{2h^2}\right)\right] \mathrm{Re}\,(\tilde{\nu}) \tag{17}$$
with the aid of (9) shows that the linear aerodynamic force adds to the damping for low q, while adding to the energy of the reed motion for large q. A detailed discussion of (17) is presented in SWB.

In order to discuss effectively the remaining term in (16), a new dimensionless quantity is introduced that eliminates the amplitude of vibration from the right-hand side of the expression for the nonlinear dimensionless force. For simplicity, this quantity is labelled as F, and is written as

$$\begin{aligned} F &= \frac{\delta p_2}{\rho Q_0 \dot{a}/2la_0}\,\frac{a_0^2}{a_1^2} \\ &= -\frac{2}{\pi}\left[3+\ln\left(\frac{l^2}{2h^2}\right)\right] \mathrm{Re}\,(\tilde{\alpha}_{1,1}+\tilde{\beta}_{1,1}) + \frac{a_0 q}{\pi h}\,\mathrm{Im}\,[2(\tilde{\alpha}_{1,1}+\tilde{\beta}_{1,1}) + \tilde{\nu}^*\tilde{\alpha}_{1,2} + 2\tilde{\nu}\tilde{\mu}_1]. \end{aligned} \tag{18}$$

Equation (18) represents a measure, for a given flow, of the dimensionless nonlinear pressure force that acts on the reed. The regenerative growth of the oscillator motion proceeds to a finite limiting amplitude when $F > 0$, whereas the nonlinear pressure force further excites the oscillator motion when $F < 0$.

The two dimensionless pressure-force components appearing in (18) are associated with the unsteady volume flow and convective acceleration terms of (2), respectively. Nonlinear effects associated with the volume-flow displacement of the reed motion were found to be insignificant for values of the volume flow that excite the reed motion.

A plot of the nonlinear-force components of (18) and their resultant as functions of the dimensionless volume flow for a typical reed configuration is shown in figure 2 (a). (Since the resultant analytical curve to the left of its intersection with the q axis has no physical meaning and since it oscillates about the q axis in this region, this part of the curve has been omitted in figures 2 (a) and (b).) It can be seen that the unsteady nonlinear force does indeed act as a damping force when the reed is self-excited. Furthermore, figure 2 (a) shows that for small values of q the effects of unsteady volume flow are predominant, whereas the effect associated with convective momentum transport increases in magnitude as q becomes large, until it becomes the more important source of nonlinear fluid–elastic damping. In view of the nonlinear analysis, such a development is not surprising.

Attention is now turned towards determining in what way mechanisms I and II contribute to the reed motion. Upon rearranging the terms of (18) into two groups representing the effects of mechanisms I and II, respectively, the expression for F can be rewritten as

$$\begin{aligned} F = &\left\{-\frac{2}{\pi}\left[3+\ln\left(\frac{l^2}{2h^2}\right)\right] \mathrm{Re}\,(\tilde{\alpha}_{1,1}) + \frac{a_0 q}{\pi h}\,\mathrm{Im}\,(2\tilde{\alpha}_{1,1})\right\} \\ &+ \left\{-\frac{2}{\pi}\left[3+\ln\left(\frac{l^2}{2h^2}\right)\right] \mathrm{Re}\,(\tilde{\beta}_{1,1}) + \frac{a_0 q}{\pi h}\,\mathrm{Im}\,[2\tilde{\beta}_{1,1} + \tilde{\nu}^*\tilde{\alpha}_{1,2} + 2\tilde{\nu}\tilde{\mu}_1]\right\}. \end{aligned} \tag{19}$$

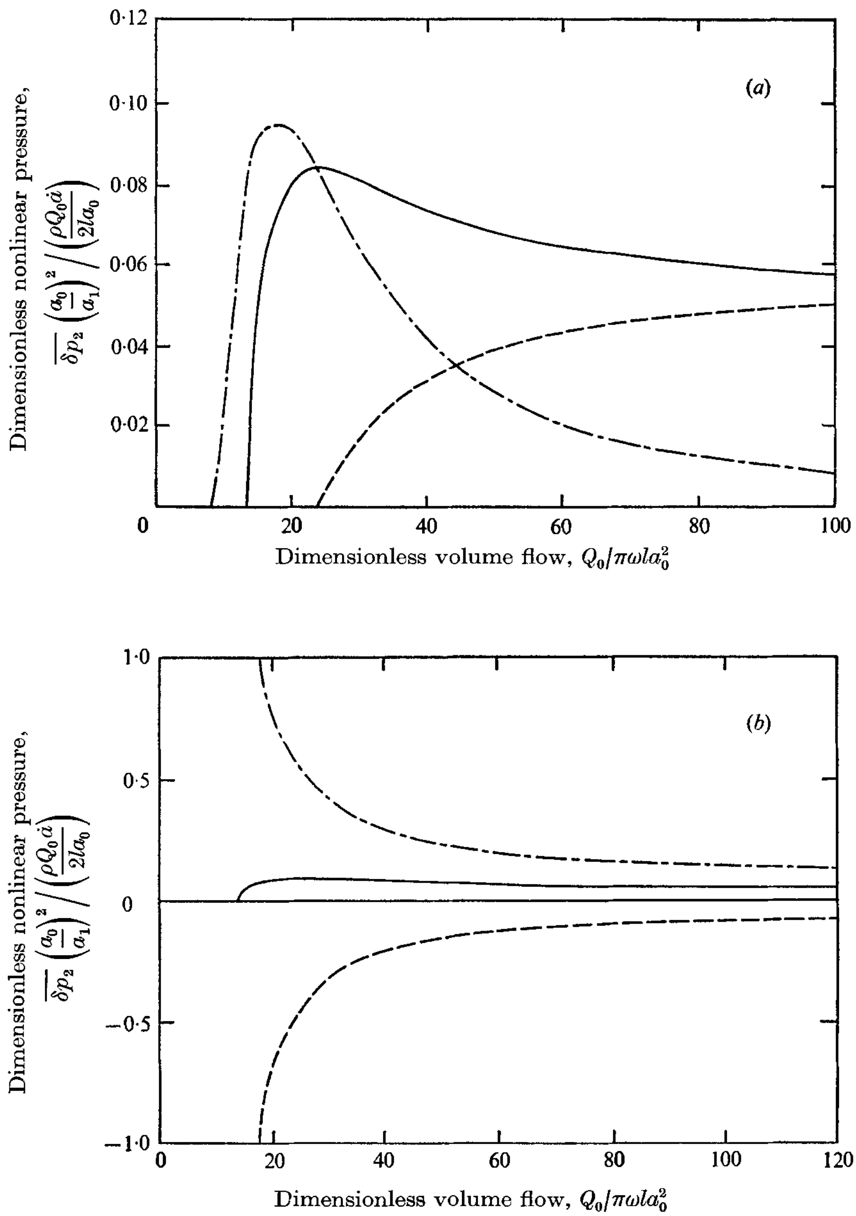

FIGURE 2. Dimensionless nonlinear pressure force as a function of dimensionless volume flow for reed B. ——, resultant nonlinear pressure force. (a) —·—, unsteady flow component; – – –, convective acceleration component. (b) —·—, contribution from mechanism II; – – –, contribution from mechanism I.

A plot of each of these components, along with their resultant, as functions of the dimensioness volume flow is shown in figure 2 (b) for the same reed configuration as in figure 2 (a). Figure 2 (b) demonstrates that the nonlinear damping force is a consequence of the interharmonic coupling previously described and represented by mechanism II. On the other hand, it is quite clear that, by itself, mechanism I would further excite the reed motion beyond the exponential rate predicted by linear analysis.

3.4. *Comments*

The approach that has been employed for showing that the net nonlinear force acting on the reed (as a consequence of higher-order disturbances in the surrounding fluid) is dissipative has allowed the revelation of some of the finer but important details of the problem. In addition to having shown that the unsteady and convective components of the flow each contribute significantly to the net nonlinear dissipative force, it has been demonstrated that, for the case of the harmonium reed, the second-harmonic and steady-state terms of the flow disturbance (which are a consequence of the self-interaction of the fundamental component) are clearly necessary to create the situation where a sufficient amount of energy is drained from the oscillator motion at finite amplitudes. Within the present theoretical framework, this drain of energy prevents the reed from hypothetically reaching catastrophic amplitudes of vibration. Furthermore, it has been shown that, in order for the energy drain just described to overcome effectively the regenerative amplitude growth predicted by linear analysis, it must first overcome a source of nonlinear excitation that is due entirely to the self-interaction of the fundamental component of the flow disturbance, as well as that due to its interaction with the oscillator motion itself.

It is important to note that the effects of mechanisms I and II taken separately are considerably larger in magnitude than their combined value. This observation is an indication of the sensitivity of the analysis to the assumptions that are made in modelling the problem; that is, the successful outcome of the theory is contingent on the degree of care exercised in formulating the assumptions that are associated with the analysis. Therefore, in order to be successful, it behoves the future investigator to exercise great care in the assumptions he uses with respect both to constructing the flow potential and to neglecting terms as the analysis proceeds.

4. Comparison of the analysis with the experiment

A comparison between the analysis and the experimental data can be achieved most conveniently by computing an effective value of the nonlinear pressure force $\overline{\delta p_2}$ from the experimental data and comparing this with the value given by equation (18). The experimental value of $\overline{\delta p_2}$ is obtained by computing the nonlinear damping force which is required to offset the observed rate of change of energy produced by the linear regenerative pressure force $\overline{\delta p_1}$. This is accomplished by noting that, when the reed reaches its limiting amplitude, the net work done by all of the forces that are in phase with the reed velocity equals zero. Therefore, the equation giving the nonlinear pressure force is

$$\frac{1}{E}\int_S (\delta p)\,\dot{a}\,dS - \left(\frac{\dot{E}}{E}\right)_{Q=0} = 0, \tag{20}$$

where the integral is taken over the surface of the reed. The second term in (20) represents the fractional rate of change of reed energy E due to material dissipation, which is assumed to be given by the damping at zero flow.

The expression that is used for computing the experimental value of $\overline{\delta p_1}$ was derived in SWB by noting that the work done by the linear pressure forces is equal to the observed energy change minus the energy change due to material dissipation. By accounting for the fact that the mode shape is nearly the same for the reeds loaded with a mass at the tip as for the uniform reed, the approximate expression giving $\overline{\delta p_1}$ was found to be

$$\frac{\overline{\delta p_1}}{\rho Q_0 \dot{a}/2la_0} = -\frac{\rho_B}{\rho}\frac{t}{a_0}\frac{4(\epsilon-\epsilon_{Q=0})}{\pi q}\left(\frac{\omega_0}{\omega}\right)^2, \tag{21}$$

which can be used for both loaded and unloaded reeds. In (21), ρ_B is the density of the reed, t is the reed thickness, ω_0 is the frequency of vibration of the unloaded reed, and ω is the frequency of vibration of the reed either loaded or unloaded by a mass at its tip. The exciting factor ϵ is a measure of the observed fractional energy change undergone by the vibrating reed, and is written in terms of the energy change as

$$\epsilon = \dot{E}/2\omega E,$$

while $\epsilon_{Q=0}$ is the damping factor measured at zero flow.

If, as indicated by analysis, $\overline{\delta p_2}$ is proportional to $a_1^2\dot{a}$, then $\overline{\delta p_2}/a_1^2\dot{a}$ can be taken out of the integral in (20). It follows that a combination of (16), (20) and (21) yields the following relationship for experimental values of F:

$$F = 4\epsilon\frac{\rho}{\rho_B}\frac{ta_0}{\pi q}\left(\frac{\omega_0}{\omega}\right)^2\int_0^l \dot{a}^2\,d\xi\bigg/\int_0^l a_1^2\dot{a}^2\,d\xi, \tag{22}$$

where l is the length of the reed. By assuming that the reed material is linearly elastic, and introducing an approximate expression based on Rayleigh's principle for the fundamental mode shape of the uniform cantilevered beam, (22) may be written in approximate form in terms of the measured surface strain S_m as

$$F = \frac{17\cdot 7t^3a_0}{S_m^2 l^4\pi q}\left(\frac{\rho_B}{\rho}\right)\left(\frac{\omega_0}{\omega}\right)^2\epsilon. \tag{23}$$

The numerical coefficient appearing in the above expression reflects the situation where the actual values of the strain are monitored and recorded near the root of the reed corresponding to the point of maximum strain. The quantities ϵ and S_m were obtained by inspection of a series of oscillograms that were taken to record the strain-gauge output of actual self-excited reeds for several values of the volume flow.

The experimental and analytical values of $\overline{\delta p_2}$ for two reeds are shown in figures 3(*a*) and (*b*). It can be seen that plotting the dimensionless nonlinear pressure force F against the dimensionless volume flow q successfully brings the experimental data for different frequencies (corresponding to various loads at the tips of the reeds) close to a common curve for each reed. These figures show that the order of magnitude of the experimentally determined nonlinear pressure

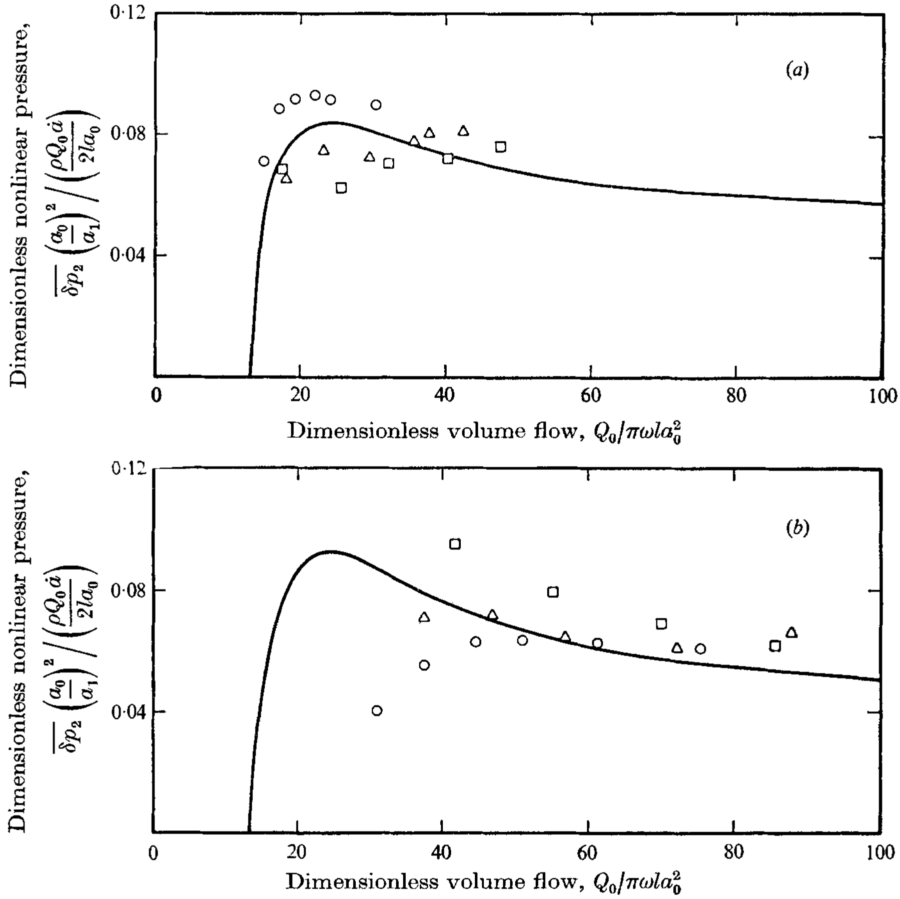

FIGURE 3. Dimensionless nonlinear pressure force as a function of dimensionless volume flow. ——, equation (18). (*a*) Reed *A*. *f* (Hz): ○, 165; △, 137; □, 120. (*b*) Reed *B*. *f* (Hz): ○, 225; △, 190; □, 161.

force is in good agreement with the results predicted by the analysis. Although there is considerable scatter in the experimental data, it is noted that, at least in figure 3 (*b*), the qualitative features of the shapes of the experimental trend and the analytical curve are similar.

A close inspection of the oscillograms revealed that the reeds did not vibrate symmetrically about their initial position of zero displacement (this position being determined just prior to the onset of excitation). The asymmetric behaviour of the reeds can be explained in part by the drifting of the limit-cycle centre as the amplitude of vibration grows. Figure 4 depicts schematically a typical oscillogram of the dynamic response envelope of a reed. Superimposed on this figure is a sketch of what the limit-cycle centre drift might look like as a function of time. This drift occurs because, in reality, the damping function is asymmetric. This asymmetric behaviour of the reed is a result of the higher-order non-oscillatory effects that contribute to the damping function. Two possible main sources of these non-oscillatory inputs are (i) nonlinear aerodynamic drag and (ii) the self-interaction of the first-harmonic component of the flow disturbance.

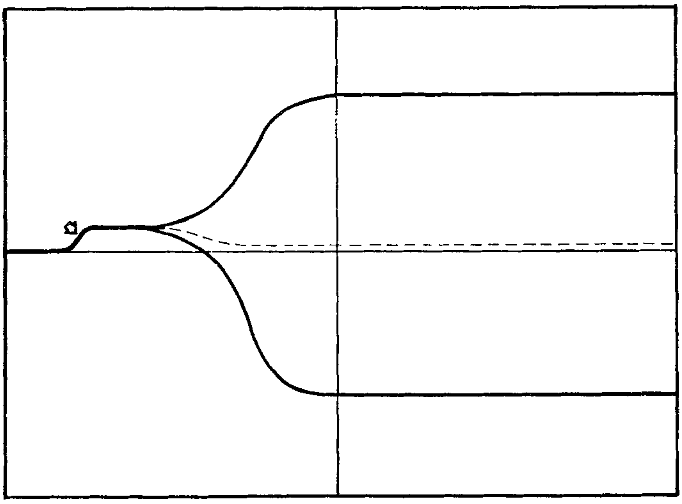

FIGURE 4. Schematic diagram of a typical oscillogram showing the growth of the vibration amplitude (solid lines) and the drift (slightly exaggerated) of the limit-cycle centre (broken line). The arrow at the left indicates the point in time of the application of constant volume flow Q_0.

These effects were not accounted for in the present analysis since interest was focused only on those contributions to the damping term that are in phase with the reed velocity.

Ideally, the nonlinear non-oscillatory inputs just described simply relocate the centre of vibration about which the reed vibrates symmetrically. The symmetry of vibration is retained because these nonlinear effects are constant during each cycle of the oscillation. Actually, however, because the position of the reed changes with respect to the location of the shallot plane during a cycle of oscillation, thereby continuously changing the system geometry, it is expected that the actual nonlinear non-oscillatory aerodynamic drag force varies slightly in magnitude, according to the position of the reed relative to the shallot plane. Consequently, this nonlinear non-oscillatory input causes the reed to vibrate asymmetrically about the drifting limit-cycle centre. Because of this irregularity, there was a built-in uncertainty in visually estimating ϵ and S_m from the oscillograms. By coupling this with the fact that F is proportional to ϵ/S_m^2, which tends to magnify the error inherent in ϵ and S_m, it is easy to see that the scatter appearing in figures 3 (*a*) and (*b*) is practically unavoidable.

5. Discussion

5.1. *Comments on assumptions*

Throughout the analysis of the present problem, the effects of viscosity were tacitly ignored. The neglect of frictional losses was justifiable, and was based on extensive flow-visualization studies (discussed by SWB; Beavers, St Hilaire & Wilson 1972), as well as on measured values of the Reynolds number corresponding to values of the volume flow for which the reed becomes excited. Values of the Reynolds number were found by experiment to be of the order of

10^3 (Beavers *et al.* 1972), where the Reynolds number is defined by $Q/l\nu$ (Q/l being the volume flow per unit length of the reed). This conveniently allowed the use of the unsteady Bernoulli equation for developing (1) and (2), upon which the subsequent analysis of §3 depended.

Considering the assumption of irrotationality and the other simplifying assumptions that were stated in the earlier sections of this paper, the comparison between analysis and experiment was surprisingly good. This fact provides incentive for investigating other seemingly complicated flow–structure interaction problems where analysis of the finite amplitude response is of interest. Because of this simplicity, given that an analysis of a particular problem of interest is successful, it is possible to generate a functional dependence of the nonlinear forces on the geometry and volume flow, thus allowing a means, via parametric studies, of investigating the possibility of discouraging or enhancing the response of a structure to an excitative force. In other words, the ability to state a functional expression provides the option of an *a priori* design approach, as opposed to an empirical trial-and-error technique. This would especially be useful in vibration and, of course, noise control.

5.2. *Natural extensions of the theory*

In the foregoing analysis, concern was focused on that part of the interactive forces that influences the structure's behaviour through the damping term. In reality, there are additional effects of the flow-field disturbances on the behaviour of the structure, but these are through the inertial and stiffness elements of the vibrator. For the case of the harmonium reed, it was seen fit to ignore these effects by retaining only that part of the resulting pressure force that was in phase with the reed velocity. If the ideas developed in this work are to be applied to other flow–structure systems, it may be necessary to retain these effects, as, for example, in the case of oscillating cylinders that are excited by vortex shedding (see, for example, Currie *et al.* 1972), to determine to what extent the flow–structure interaction of the type under discussion contributes to the observed soft-spring distortion of the amplitude–frequency response of some structures. (Other structures may experience hard-spring distortions.) This distortion is manifested by a shift in the location of resonance to the left (for the hard-spring case, the shift is to the right), thereby creating a situation where double-valued amplitude responses are possible. For the case of the harmonium reed, the maximum detectable shift was about one cycle per second, or $\Delta f/f_0 < 0{\cdot}01$.

Additional information provided by this theory helps to explain, in part, the source of asymmetry in the damping function. Asymmetric behaviour of the reeds was observed (§4), and its extent (as an effect of flow-field disturbances) in this aspect of the structure's behaviour can be estimated by retaining the second-order contributions to the damping function.

It is interesting to note that these extensions as applied to the case of the harmonium reed are a by-product of the foregoing analysis and that, therefore, no complications due to the analysis of additional terms in (6) would be necessary.

5.3. *Relevance of acoustic radiation*

The radiation of sound from a self-excited structure is a phenomenon which is interesting from many points of view. The musician, for example, is interested in the sound radiated from reeds because it is an essential datum for the design of a musical system. The noise-control expert in industry, on the other hand, is interested in the sound radiation as a means of assessing the annoyance that flow-induced structural vibrations can cause.

Now that the harmonium-reed source has been described from both a monopole point of view (fluctuating flow) and a dipole point of view (fluctuating forces at the reed surface), it would seem that the next step naturally would be to attempt analytically to construct the acoustic signal that emanates from this source. (Such an analysis would, of course, have to include effects of environmental influences that would modify the signal, such as baffles and the room characteristics.) It has already been demonstrated in the laboratory, though not analytically proved, that the amplitudes of the acoustical and dynamical signals of the reed grow in a similar manner (St Hilaire 1970). This would seem to indicate that the fraction of energy that is dissipated for the purpose of producing sound is either constant or very nearly so, as the signal grows.

Once a quasi-steady analysis of the acoustic field were realized, then by making use of the experimental evidence that the dynamical and acoustical signals for reeds are characteristically similar, a time-dependent development of the acoustic disturbance would be established empirically. The knowledge gained from such a study could have long-range use with respect to other flow–structure interaction systems that are capable of radiating a perceptible signal.

An applied mechanician could find the data on the sound radiation interesting from another point of view, for it could in turn lead to diagnostic clues about the emitting mechanism itself. This was indeed the case for the investigation that led to this paper. The successful outcome of this paper demonstrates the promise of acoustics as a diagnostic tool for investigating complicated sound-radiating flow systems.

The clue which gave rise to the analytical approach outlined in this paper came about when it was noted that a reed vibrating in simple harmonic motion could radiate a multi-harmonic acoustic signal. By making use of the implied multi-harmonic content of the flow-field disturbance, it was possible to describe the acoustic source fluid dynamically in some detail.

The use of an acoustic signal as a diagnostic tool to help identify certain characteristics of complicated sound sources associated with flow–structure interactions is under current investigation.

6. Conclusions

Upon neglecting nonlinear energy-dissipating forces which arise from sources such as material damping and large amplitude aerodynamic drag, it was found that the mechanism responsible for inhibiting the indefinite growth of the harmonium reed's amplitude of vibration may be attributed to the existence of the higher-order unsteady flow perturbations which are induced by the reed motion itself. More specifically, the mechanism responsible for the nonlinear energy drain from the reed motion was found to be a result of the mutual interaction of the first-harmonic component with the steady-state and second-harmonic components of the flow disturbance. This mutual interaction results in nonlinear forces which act on the reed so as to slow down the rate of amplitude growth until a stable situation arises; that is, until a limiting amplitude is reached by the reed. In addition, it was established that both the unsteady and convective acceleration terms of the unsteady Bernoulli equation contribute significantly to the nonlinear force that acts on the structure.

The relative simplicity of this theory makes it a desirable tool for analysing other systems in flow–structure interactions research which appear to have a questionable source of nonlinear dissipation. The advantage of this type of analysis lies in the fact that the flow about the oscillating structure may be assumed to be potential if certain conditions about the flow are satisfied. Once an appropriate flow field has been determined and account of the unsteady higher-order effects has been made, the analysis should proceed in much the same way as in the special case of the harmonium reed. Although such an analysis might not explain the mechanism of excitation as it did for the case of reeds found in musical instruments (SWB), the subsequent results arising from the nonlinear aspects of the analysis should, nevertheless, be applicable for determining at least one of several possible sources of nonlinear energy dissipation for large amplitude motion, regardless of the mechanism of excitation.

The authors gratefully acknowledge help from the technical staff at Aerodyne Research, Inc., Burlington, Massachusetts, for preparing the final manuscript. This research was supported by the Lufkin Post-Doctoral Fellowship awarded by the Department of Mechanical Engineering, Tufts University.

REFERENCES

Baker, W. E., Woolam, W. E. & Young, D. 1967 Air and internal damping of thin cantilever beams. *Int. J. Mech. Sci.* **9**, 743–760.

Beavers, G. S., St Hilaire, A. O. & Wilson, T. A. 1972 Vortex growth in two-dimensional coalescing jets. *J. Basic Engng*, D **94**, 500–503.

Crandall, S. H., Khabbaz, G. R. & Manning, J. E. 1964 Random vibration of an oscillator with nonlinear damping. *J. Acoust. Soc. Am.* **36**, 1330–1334.

Currie, I. G., Hartlen, R. T. & Martin, W. W. 1972 The response of circular cylinders to vortex shedding. *Proc. Symp. on Flow-Induced Structural Vibrations IUTAM/IAHR*, Karlsruhe, Germany (tech. session B).

Lazan, B. J. & Goodman, L. E. 1961 Material and interface damping. In *Shock and Vibration Handbook* (ed. C. M. Harris & C. E. Crede), vol. 2, chap. 36. McGraw-Hill.

St Hilaire, A. O. 1970 Fluid dynamic excitation of the organ reed. Ph.D. thesis, University of Minnesota.

St Hilaire, A. O. 1971 An approximation of the pressure field induced by an oscillating plane bounded by two finite parallel line sinks of oscillating width. *Dept. Mech. Engng, Tufts University, Rep.* TUMER 71-1.

St Hilaire, A. O. 1972 Analysis of the forces acting on a self-sustained oscillating reed-like structure coupled with a flowing fluid. *Dept. Mech. Engng, Tufts University, Rep.* TUMER 72-1.

St. Hilaire, A. O., Wilson, T. A. & Beavers, G. S. 1971 Aerodynamic excitation of the harmonium reed. *J. Fluid Mech.* **49**, 803–816.